Crystal isomorphisms and wall-crossing maps for rational Cherednik algebras

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CRYSTAL ISOMORPHISMS AND WALL CROSSING MAPS FOR RATIONAL CHEREDNIK ALGEBRAS

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Abstract. We show that the wall crossing bijections between simples of the category $\mathcal{O}$ of the rational Cherednik algebras reduce to particular crystal isomorphisms which can be computed by a simple combinatorial procedure on multipartitions of fixed rank.

1. Introduction

The rational Cherednik algebra associated to a complex reflection group W was introduced by Etingof and Ginzburg in [1] as a particular symplectic reflection algebra. In this paper, we will focus on the case where W is the complex reflection group $G(l,1,n) := (\mathbb{Z}/l\mathbb{Z})^n \rtimes \mathfrak{S}_n$. The rational Cherednik algebra then depends on the choice of a certain parameter s in a $(l+1)$-dimensional $\mathbb{C}$-vector space. There is a distinguished category of modules over these algebras, the category $\mathcal{O}$, which may be constructed in the same spirit as the BGG category for a reductive Lie algebra. This category has been intensively studied during the last decade because of its interesting structure and also its connection with other important mathematical objects such as cyclotomic Hecke algebras or cyclotomic q-Schur algebras.

As the simple modules of the associated complex reflection groups, the simple modules in this category are labelled by the set of multipartitions. A natural and important question is then to understand how the set of simple modules in the category $\mathcal{O}$ for different choices of the parameter s are related. In [9], Losev has defined a collection of hyperplanes in the space of parameters called "essential walls". Then, he has shown the existence of a perverse and derived equivalence between the categories $\mathcal{O}$ associated to distinct parameters s and s' with integral difference separated by a single wall. In particular, this equivalence induces a bijection between the simples of these categories. These bijections, called "wall crossing maps", are of great interest because they commute with both actions of the Heisenberg algebra and the affine type A algebra $\mathfrak{g}$. In particular, they can be used to compute the support (as coherent sheaves on the reflection representation of W) of the simple modules in the category $\mathcal{O}$ and to obtain a classification of the finite-dimensional irreducible representations. Indeed, by the works of Losev [9], in the case where the parameter s satisfies a certain "asymptotic" condition, the support can be quite easily computed. Now, one can

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always go from a category $\mathcal{O}$ parametrized by a parameter s to a category $\mathcal{O}$ parametrized by an asymptotic parameter by using a composition of these wall-crossing maps. As, in addition, they leave the support invariant, these wall-crossing maps thus allow its computation in general.

The goal of the present paper is to show that, despite their very abstract definition, there is a simple combinatorial procedure to compute the wall crossing bijections.

Because of the above properties, these bijections can also be interpreted as crystal isomorphisms for certain integrable $\mathfrak{g}$-modules: the Fock spaces. On the other hand, in [6], we described distinguished crystal isomorphisms between such Fock spaces and presented a simple procedure on multipartitions of fixed rank to compute them. The aim of this paper is to show how these two isomorphisms are related. To do this, we first review the Uglov $\mathfrak{g}$-module structure of the Fock space and explain how it can be extended. We then construct and explicitly describe crystal isomorphisms corresponding to this generalized structure. The last section explores the connections of these isomorphisms with Losev's wall crossing bijections. We explain how the crystals appearing in the context of Cherednik algebras are related with the usual crystals of Fock spaces (in the sense of Uglov). Finally, the main result gives a simple combinatorial procedure on l-partitions of fixed rank for computing an arbitrary wall crossing bijection without referring to any crystal structure

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2. Colored graphs and Fock spaces of JMMO type

We first describe structures of colored oriented graphs on the sets of l-partitions (that we define below), then focus on a particular case which is connected to the action of the quantum affine group $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$.

2.1. Combinatorics of l-partitions.

Definition 2.1. *Let $n \in \mathbb{N}$ and $l \in \mathbb{N}$:*

- *A partition λ of rank n is a sequence*

$$(\lambda_1, \dots, \lambda_r),$$

of decreasing non negative integers such that

$$\sum_{1 \leq i \leq r} \lambda_i = n.$$

By convention, we identify two partitions which differ by parts equal to 0.

- *A l-partition (or multipartition) $\boldsymbol{\lambda}$ of rank n is an l-tuple of partitions $(\lambda^1,\dots,\lambda^l)$ such that the sum of the rank of the partitions λ^i for $1\leq i\leq l$ is n. We denote by $\Pi^l(n)$ the set of all l-partitions of rank n and if $\boldsymbol{\lambda}\in\Pi^l(n)$, we sometimes write $\boldsymbol{\lambda}\vdash_l n$. The empty l-partition (which is the l-tuple of empty partitions) is denoted by $\boldsymbol{\emptyset}$.*

Let $\kappa\in\mathbb{Q}_+$ and $\mathbf{s}=(s_1,s_2,\dots,s_l)\in\mathbb{Q}^l$. We set $s:=(\kappa,\mathbf{s})$.
One can associate to each $\boldsymbol{\lambda}\vdash_l n$ its *Young diagram*:

$$[\boldsymbol{\lambda}]=\{(a,b,c)\ a\geq 1,\ 1\leq c\leq l, 1\leq b\leq \lambda_a^c\}.$$

This diagram sometimes will be identified with the l-partition itself. We define the *s-content* of a node $\gamma=(a,b,c)\in[\boldsymbol{\lambda}]$ as follows:

$$\mathrm{cont}(\gamma)=b-a+s_c\in\mathbb{Q},$$

and the *residue* of γ is by definition the content of the node in $\mathbb{Q}/\kappa^{-1}\mathbb{Z}$ when $\kappa\neq 0$ and equals the residue when $\kappa=0$.

Definition 2.2. *Let I_s be the subset of $\mathbb{Q}/\kappa^{-1}\mathbb{Z}$ formed by the classes $x+s_j+\kappa^{-1}\mathbb{Z}$ where $x\in\mathbb{Z}$ and $j\in\{1,\dots,l\}$ when $\kappa\neq 0$ and the subset of $\mathbb{Q}$ formed by the rationals of the form $x+s_j$ where $x\in\mathbb{Z}$ and $j\in\{1,\dots,l\}$ when $\kappa=0$.*

We say that γ is a z-node of $\boldsymbol{\lambda}$ when $\mathrm{res}(\gamma)=z+\kappa^{-1}\mathbb{Z}$. Finally, we say that γ is *removable* when $\gamma=(a,b,c)\in\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}\backslash\{\gamma\}$ is an l-partition. Similarly γ is *addable* when $\gamma=(a,b,c)\notin\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}\cup\{\gamma\}$ is an l-partition.

Fix $z\in I_s$. We assume that we have a total order $\leq$ on the set of z-nodes of an arbitrary l-partition. We then define two operators depending on z as follows. We consider the set of addable and removable z-nodes of our l-partition. Let $w_z(\boldsymbol{\lambda})$ be the word obtained first by writing the addable and removable z-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $\leq$ next by encoding each addable z-node by the letter A and each removable i-node by the letter R. Write $\widetilde{w}_z(\boldsymbol{\lambda})=A^pR^q$ for the word derived from w_z by deleting as many subwords of type RA as possible. The word $w_z(\boldsymbol{\lambda})$ is called the z-word of $\boldsymbol{\lambda}$ and $\widetilde{w}_z(\boldsymbol{\lambda})$ the reduced z-word of $\boldsymbol{\lambda}$. The addable z-nodes in $\widetilde{w}_z(\boldsymbol{\lambda})$ are called the *normal addable z-nodes*. The removable z-nodes in $\widetilde{w}_z(\boldsymbol{\lambda})$ are called the *normal removable z-nodes*. If $p>0$, let γ be the rightmost addable z-node in $\widetilde{w}_z$. The node γ is called the *good addable z-node*. If $q>0$, the leftmost removable i-node in $\widetilde{w}_z$ is called the *good removable z-node*.

We then define $\widetilde{e}_z^{\leq}\boldsymbol{\mu}=\boldsymbol{\lambda}$ and $\widetilde{f}_z^{\leq}\boldsymbol{\lambda}=\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by adding to $\boldsymbol{\lambda}$ a good addable z-node, or equivalently, $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\mu}$ by removing a good removable z-node. If $\boldsymbol{\mu}$ as no good removable z-node then we set $\widetilde{e}_z^{\leq}\boldsymbol{\mu}=0$ and if $\boldsymbol{\lambda}$ has no good addable z-node we set $\widetilde{f}_z^{\leq}\boldsymbol{\lambda}=0$.

2.2. **Extended JMMO structure associated to s.** We can define from s and $\leq$ a colored oriented graph $\mathcal{G}_{s,\leq}$ as follows:

- vertices : the l-partitions $\boldsymbol{\lambda}\vdash_l n$ with $n\in\mathbb{Z}_{\geq 0}$

- the arrows are colored by elements in I_s and we have: $\boldsymbol{\lambda} \overset{i}{\to} \boldsymbol{\mu}$ for $i \in \mathbb{Q}/\kappa^{-1}\mathbb{Z}$ if and only if $\widetilde{e}_i^{\leq}\boldsymbol{\mu} = \boldsymbol{\lambda}$, or equivalently $\widetilde{f}_i^{\leq}\boldsymbol{\lambda} = \boldsymbol{\mu}$.

The l-partitions such that $\widetilde{e}_z\boldsymbol{\mu} = 0$ for all z will be called *highest weight vertices*. The set I_s of elements in $\mathbb{Q}/\kappa^{-1}\mathbb{Z}$ coloring the arrows is called the indexing set of the graph. Observe the graph $\mathcal{G}_{s,\leq}$ is not an affine $A^{(1)}_{e-1}$-crystal graph in general notably because its indexing set can be distinct from $\mathbb{Z}/e\mathbb{Z}$.

Finally two graphs $\mathcal{G}_{s_1,\leq_1}$ and $\mathcal{G}_{s_1,\leq_2}$ on l-partitions with indexing sets I_1 and I_2 are isomorphic if there exists a bijection

$$\Psi : \Pi^l(n) \to \Pi^l(n), \tag{1}$$

and a bijection:

$$\psi : I_1 \to I_2, \tag{2}$$

such that

- $\boldsymbol{\lambda}$ is a highest weight vertex in $\mathcal{G}_{s_1,\leq_1}$ if and only if $\Psi(\boldsymbol{\lambda})$ is a highest weight vertex in $\mathcal{G}_{s_2,\leq_2}$,
- we have an arrow $\boldsymbol{\lambda} \overset{i}{\to} \boldsymbol{\mu}$ in $\mathcal{G}_{s_1,\leq_1}$ if and only if we have an arrow $\Psi(\boldsymbol{\lambda}) \overset{\psi(i)}{\to} \Psi(\boldsymbol{\mu})$ in $\mathcal{G}_{s_2,\leq_2}$.

In addition, if Ψ is the identity, we will say that the two graphs are equivalent (in particular, the graph structures coincide up to their coloring). We will see that, for a good choice of the orders $\leq_1$ and $\leq_2$, the graphs $\mathcal{G}_{s_1,\leq_1}$ and $\mathcal{G}_{s_2,\leq_2}$ have the structure of a Kashiwara crystal graph. In this case, a crystal isomorphism between $\mathcal{G}_{s_1,\leq_1}$ and $\mathcal{G}_{s_2,\leq_2}$ is a graph isomorphism such that $\psi = id$. In particular, each crystal isomorphism yields a graph isomorphism but the converse is false in general.

2.3. **Extended JMMO Fock space structure.** We now define a Fock space structure that, as far as we know, first appeared in the work of Gerber [3]. This structure generalizes the ones defined by Uglov [12] and Jimbo-Misra-Miwa-Okado (JMMO) [8].

Condition 2.3. *We assume in this section that* $\mathbf{s} = (s_1, \ldots, s_l) \in \mathbb{Z}^l$ *and* $\kappa = 1/e$ *where* $e \in \mathbb{N}_{>1} \sqcup \{\infty\}$ *(with* $\kappa = 0$ *when* $e = \infty$*).*

Let $\mathfrak{g}_e := \mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ be the quantum group of affine type $A^{(1)}_{e-1}$ if e is finite, otherwise we set $\mathfrak{g}_\infty := \mathcal{U}_q(\mathfrak{sl}_\infty)$. The associative $\mathbb{Q}(q)$-algebra $\mathfrak{g}_e$ has generators e_i, f_i, t_i, t_i^{-1} (for $i = 0, ..., e-1$) and ∂. We refer the reader to [12, §2.1] for the relations they satisfy (we do not use them in the sequel.) We denote by $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$ the subalgebra generated by e_i, f_i, t_i, t_i^{-1} (for $i = 0, ..., e-1$). We write $\Lambda_i, i = 0, ..., e-1$ for the fundamental weights of $\mathfrak{g}_e$.

The *Fock space* $\mathcal{F}$ is the $\mathbb{Q}(q)$-vector space defined as follows:

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\boldsymbol{\lambda} \vdash_l n} \mathbb{Q}(q)\boldsymbol{\lambda}.$$

Set $I=\mathbb{Z}$ if $e=\infty$ and $I=\mathbb{Z}/e\mathbb{Z}$ otherwise.

For any $e\in\mathbb{N}_{>1}\sqcup\{\infty\}$, there is an action of $\mathfrak{g}_e$ on the Fock space $\mathcal{F}$. This action can be regarded as a generalization of Uglov's construction [12, §2.1]. It is defined by using an order on the nodes on the l-partitions with the same residue modulo e. This order depends on the choice of an l-tuple of rational numbers $\mathbf{m}=(m_1,\dots,m_l)$. If e is finite, for $1\leq i,j\leq l$ with $i\neq j$ and $N\in\mathbb{Z}$, let us define

$$\mathfrak{m}^{\mathbf{s},e}_{i,j,N}:=\{(m_1,\dots,m_l)\in\mathbb{Q}^l\ |s_i-m_i-(s_j-m_j)=N.e\}.$$

If $e=\infty$, we define for $1\leq i,j\leq l$

$$\mathfrak{m}^{\mathbf{s},\infty}_{i,j}:=\{(m_1,\dots,m_l)\in\mathbb{Q}^l\ |s_i-m_i-(s_j-m_j)=0\}.$$

Let $\mathfrak{M}^{\mathbf{s},e}$ be the union of the hyperplanes $\mathfrak{m}^{\mathbf{s},e}_{i,j,N}$ for all $1\leq i,j\leq l$ and $N\in\mathbb{Z}$ if e is finite and $\mathfrak{m}^{\mathbf{s},\infty}_{i,j}$ for all $1\leq i,j\leq l$ if $e=\infty$. Now consider $\mathbf{m}\notin\mathfrak{M}^{\mathbf{s},e}$. Let γ, γ' be two removable or addable i-nodes of $\boldsymbol{\lambda}$ for $i\in I_s$. We denote

$$\gamma\preceq_{\mathbf{m}}\gamma'\overset{\text{def}}{\Longleftrightarrow}b-a+m_c<b'-a'+m_{c'}. \tag{3}$$

Thanks to our assumption $\mathbf{m}\notin\mathfrak{M}^{\mathbf{s},e}$, it is easy to verify that the above definition indeed defines a total order on the set of i-nodes of any l-partition. This thus yields a graph $\mathcal{G}_{e,\mathbf{m},\mathbf{s}}$ as in §2.2. It was also proved in [3] that one can mimic Uglov's Fock space construction and define a $\mathfrak{g}_e$-action on the Fock space $\mathcal{F}$ from any order $\preceq_{\mathbf{m}}$. This gives an integrable $\mathfrak{g}_e$-module that we denote by $\mathcal{F}_{e,\mathbf{m},\mathbf{s}}$. The submodule generated by the empty l-partition is then an irreducible highest weight module of weight $\Lambda_{\mathbf{s}}=\Lambda_{s_1}+\cdots+\Lambda_{s_l}$.

Remark 2.4. *Note that the Fock space $\mathcal{F}_{e,\mathbf{m},\mathbf{s}}$ depends on the choice of $\mathbf{m}$ (because of the order $\preceq_{\mathbf{m}}$) and on the choice of $\mathbf{s}$ modulo e (because of the definition of the residue of a node). Also in the case where $\mathbf{s}=(s_1,\dots,s_l)$ and $\mathbf{s}'=(s'_1,\dots,s'_l)$ satisfy $s_j\equiv s'_j$ (mod e) for $j=1,\dots,l$, the associated Fock spaces can be identified and we can write $\mathcal{F}_{e,\mathbf{m},\mathbf{s}}=\mathcal{F}_{e,\mathbf{m},\mathbf{s}'}$.*

Remark 2.5. *The inverse order $\preceq^{-}_{\mathbf{m}}$ of $\preceq_{\mathbf{m}}$ also yields the structure of an integrable $\mathfrak{g}_e$-module on $\mathcal{F}$ we denote by $\mathcal{F}^{-}_{e,\mathbf{m},\mathbf{s}}$.*

2.4. **Relations with JMMO Fock space structure (1).** The operators $\widetilde{e}^{\preceq_{\mathbf{m}}}_{z}$ and $\widetilde{f}^{\preceq_{\mathbf{m}}}_{z}$ defined from the order $\preceq_{\mathbf{m}}$ as in §2.2 coincide in fact with the Kashiwara crystal operators and $\mathcal{G}_{e,\mathbf{m},\mathbf{s}}$ or $\mathcal{G}_{\infty,\mathbf{m},\mathbf{s}}$ are the crystal graphs corresponding to the $\mathfrak{g}_e$-module structure on our Fock space. To recover the crystal structure used by Uglov (or the crystal structure introduced by JMMO), it suffices to choose $\mathbf{m}$ such that

$$m_c=s_c+\delta_c,\ c=1,\dots,l,$$

where $e>\delta_1>\cdots>\delta_l\geq 0$. We will simply denote by $\mathcal{G}_{e,\mathbf{s}}$ this JMMO structure.

Now let us consider $\mathbf{m}' \notin \mathfrak{M}^{\mathbf{s},e}$. For $c = 1, \dots, l$, define $\delta'_c \in \mathbb{Q}$ as the unique rational number such that $0 \leq \delta'_c < e$ which is equivalent to $m'_c - s_c$ modulo e. Thus there exists $(s'_1, \dots s'_l) \in \mathbb{Z}^l$ such that $s_j \equiv s'_j (\text{mod } e)$ and

$$m'_c = s'_c + \delta'_c, \ c = 1, \dots, l.$$

We have

$$e > \delta'_{\sigma(1)} > \cdots > \delta'_{\sigma(l)} \geq 0,$$

for a permutation $\sigma \in \mathfrak{S}_l$. Then the map

$$\left\{ \begin{array}{l} \mathcal{F}_{e,\mathbf{m}',\mathbf{s}} = \mathcal{F}_{e,\mathbf{m}',\mathbf{s}'} \overset{\sigma}{\to} \mathcal{F}_{e,\sigma(\mathbf{m}'),\sigma(\mathbf{s})} \\ (\lambda^1, \dots, \lambda^l) \mapsto (\lambda^{\sigma(1)}, \dots, \lambda^{\sigma(l)}) \end{array} \right. \tag{4}$$

is an isomorphism of $\mathfrak{g}_e$-modules. It also defines a crystal isomorphism between the crystal $\mathcal{G}_{e,\mathbf{m}',\mathbf{s}}$ and the JMMO crystal $\mathcal{G}_{e,\sigma(\mathbf{s})}$.

This implies that for any $\mathbf{m}_1 \notin \mathfrak{M}^{\mathbf{s},e}$ and $\mathbf{m}_2 \notin \mathfrak{M}^{\mathbf{s},e}$, the Fock spaces $\mathcal{F}_{e,\mathbf{m}_1,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{m}_2,\mathbf{s}}$ are isomorphic. The crystals $\mathcal{G}_{e,\mathbf{m}_1,\mathbf{s}}$ and $\mathcal{G}_{e,\mathbf{m}_2,\mathbf{s}}$ are then also isomorphic as crystals. This means they are isomorphic in the sense of § 2.2 with indexing set $\mathbb{Z}/e\mathbb{Z}$ and $\psi = id$. The modules $\mathcal{F}_{e,\mathbf{m}_1,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{m}_2,\mathbf{s}}$ are reducible in general, so such an isomorphism is not unique. However, its restriction to the connected component of $\mathcal{G}_{e,\mathbf{m}_1,\mathbf{s}}$ with empty highest weight vertex yields the connected component of $\mathcal{G}_{e,\mathbf{m}_2,\mathbf{s}}$ with empty highest weight vertex.

3. Description of the canonical crystal isomorphisms

In this section we assume that $s = (\kappa, \boldsymbol{s})$ with $\kappa = \frac{1}{e}$ and $\boldsymbol{s} \in \mathbb{Z}^l$. Then $\mathcal{G}_{e,\mathbf{m},\mathbf{s}}$ is a Kashiwara crystal for any $\mathbf{m} \notin \mathfrak{M}^{\mathbf{s},e}$.

3.1. **Crystal isomorphisms.** The hyperplanes $\mathfrak{m}^{\mathbf{s},e}_{i,j,N}$ divide $\mathbb{R}^l$ into chambers. We first show that the orders $\preceq_{\mathbf{m}}$ are the same for all the parameters $\mathbf{m}$ in the same (open) chambers. We also show that one can restrict to a finite sets of chambers in order to understand our crystal isomorphisms.

Proposition 3.1. *Assume that $\mathbf{m}_1$ and $\mathbf{m}_2$ are both in the same chamber with respect to the decomposition in §2.3, then the orders $\preceq_{\mathbf{m}_1}$ and $\preceq_{\mathbf{m}_2}$ on the i-nodes of an arbitrary l-partition coincide.*

Proof. Consider $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ two distinct i-nodes and assume we have $\gamma \prec_{\mathbf{m}_1} \gamma'$ but $\gamma' \prec_{\mathbf{m}_2} \gamma$. This means that:

$$b - a + m_{1,c} < b' - a' + m_{1,c'}, \ b - a + m_{2,c} > b' - a' + m_{2,c'}$$
$$\text{and } b - a + s_c = b' - a' + s_{c'} + ke \text{ with } k \in \mathbb{Z}.$$

We get $b' - a' = b - a + s_c - s_{c'} - ke$. By replacing $b' - a'$ by its expression in terms of a and b in the first above inequality we obtain:

$$(s_c - m_{1,c}) - (s_{c'} - m_{1,c'}) > ke, \tag{5}$$

whereas the second inequality yields:

$$(s_c - m_{2,c}) - (s_{c'} - m_{2,c'}) < ke. \tag{6}$$

This means that $\mathbf{m}_1$ and $\mathbf{m}_2$ are separated by the affine hyperplane with equation $(s_c - m_c) - (s_{c'} - m_{c'}) = ke$, so we get the desired contradiction. □

Proposition 3.2. *Consider a wall* $\mathfrak{m}_{i,j,N}^{\mathbf{s},e}$ *such that:*

$$|N.e + (s_j - s_i)| \geq n,$$

and pick two parameters $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *separated by this wall (and only it) then the order* $\preceq_{\mathbf{m}_1}$ *and* $\preceq_{\mathbf{m}_2}$ *are the same on* $\Pi^l(n)$*.*

Proof. Consider $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ two distinct z-nodes in a given l-partition of n such that $\gamma \prec_{\mathbf{m}_1} \gamma'$ but $\gamma' \prec_{\mathbf{m}_2} \gamma$. Since we know that $\mathbf{m}_1$ and $\mathbf{m}_2$ are separated by the unique wall $\mathfrak{m}_{i,j,N}^{\mathbf{s},e}$, one can assume (with the notation of the previous proof) that $c = i$ and $c' = j$ and $k = N$. We obtain:

$$b - a + s_i = b' - a' + s_j + N.e,$$

but as we have

$$|(b - a) - (b' - a')| < n,$$

this leads to a contradiction. □

The above propositions shows that if we choose $\mathbf{m}_1$ and $\mathbf{m}_2$ such that :

- these parameters belong to the same chamber with respect to the decomposition in §2.3,
- or satisfy the condition of Proposition 3.2,

then the associated crystal structures are not simply isomorphic but equal. To investigate our canonical isomorphims, we thus have a finite set $\mathfrak{M}_n^{\mathbf{s},e}$ of hyperplanes and chambers to consider (depending on n), namely $\mathfrak{M}_n^{\mathbf{s},e}$ is the union of the $\mathfrak{m}_{i,j,N}^{\mathbf{s},e}$ for $1 \leq i < j \leq l$ and $|N.e + (s_j - s_i)| \leq n$. Indeed, the canonical isomorphisms we aim to characterize will be equal to the identity if we stay inside one chamber. So it just remains to understand what happens when we move from one chamber to another. This is equivalent to describing the crystal isomorphisms corresponding to the crossings of the walls $\mathfrak{m}_{i,j,N}^{\mathbf{s},e}$ with $1 \leq i < j \leq l$ and $|N.e + (s_j - s_i)| \leq n$.

3.2. **A combinatorial procedure.** We first study the case of $e = \infty$, define and describe our canonical crystal isomorphisms. The walls are then defined as:

$$\mathfrak{m}_{i,j}^{\mathbf{s},\infty} := \{(m_1, \ldots, m_l) \in \mathbb{Q}^l | s_i - m_i - (s_j - m_j) = 0\},$$

for each $1 \leq i, j \leq l$. Let us first consider the case where $l = 2$. Set $\mathbf{s} = (s_1, s_2)$. We wish to describe a crystal isomorphism when we cross a wall:

$$\mathfrak{m}_{1,2}^{(s_1,s_2),\infty} := \{(m_1, m_2) \in \mathbb{Q}^2 | s_1 - m_1 - (s_2 - m_2) = 0\}.$$

Let (λ^1, λ^2) be a bipartition of n. We define the minimal integer $d \geq |s_1 - s_2|$ such that $\lambda^1_{d+1+s_1-s_2} = \lambda^2_{d+1} = 0$ if $s_2 \geq s_1$, and otherwise the minimal integer $d \geq |s_1 - s_2|$ such that $\lambda^2_{d+1+s_2-s_1} = \lambda^1_{d+1} = 0$. To (λ^1, λ^2), we associate its symbol. This is the two row tableau:

$$S(\lambda^1,\lambda^2) = \begin{array}{|c|c|c|c|c|c|}\hline s_2-d+\lambda^2_{d+1} & \cdots & \cdots & \cdots & s_2-1+\lambda^2_2 & s_2+\lambda^2_1 \\ \hline s_2-d+\lambda^1_{d+1+s_1-s_2} & \cdots & s_1+\lambda^1_1 \\ \cline{1-3}\end{array}$$

when $s_2 \geq s_1$

$$S(\lambda^1,\lambda^2) = \begin{array}{|c|c|c|c|c|c|}\cline{1-3} s_1-d+\lambda^2_{d+1+s_2-s_1} & \cdots & s_2+\lambda^2_1 \\ \hline s_1-d+\lambda^1_{d+1} & \cdots & \cdots & \cdots & s_1-1+\lambda^1_2 & s_1+\lambda^1_1 \\ \hline\end{array}$$

when $s_2 < s_1$.

We will write $S(\lambda^1, \lambda^2) = \binom{L_2}{L_1}$ where the top row (resp. the bottom row) corresponds to λ^2 (resp. λ^1). By definition of d, the entry in the leftmost column of $S(\lambda^1, \lambda^2)$ is equal to $s_2 - d$ (resp. $s_1 - d$) when $s_2 \geq s_1$ (resp. $s_2 < s_1$). So from $S(\lambda^1, \lambda^2)$ it is easy to obtain s_1 and s_2 since $|s_1 - s_2|$ is the difference between the lengths of L_1 and L_2. Once we have $S(\lambda^1, \lambda^2)$ and (s_1, s_2), we can recover the bipartition (λ^1, λ^2).

We now recall a combinatorial procedure used in [6] to describe the combinatorial R-matrix

$$\mathcal{G}_{\infty,(s_1,s_2)} = \mathcal{G}_{\infty,s_1} \otimes \mathcal{G}_{\infty,s_2} \simeq \mathcal{G}_{\infty,s_2} \otimes \mathcal{G}_{\infty,s_1} = \mathcal{G}_{\infty,(s_2,s_1)}.$$

On the set of bipartitions, the above R-matrix can be written

$$\begin{array}{rcll} R^{\infty}_{(s_1,s_2)} & : \Pi^2(n) & \rightarrow & \Pi^2(n) \\ & (\lambda^1,\lambda^2) & \mapsto & (\widetilde{\lambda}^2, \widetilde{\lambda}^1). \end{array}$$

We proved in [6] it is a rank preserving crystal isomorphism. Also $R^{\infty}_{(s_1,s_2)}$ is the unique $\mathfrak{g}_\infty$-crystal isomorphism between the two Fock spaces $\mathcal{G}_{\infty,(s_1,s_2)}$ and $\mathcal{G}_{\infty,(s_2,s_1)}$ because there is no multiplicity into their decomposition in irreducible components (only in level 2). We also have $R^{\infty}_{(s_1,s_2)} \circ R^{\infty}_{(s_2,s_1)} = R^{\infty}_{(s_2,s_1)} \circ R^{\infty}_{(s_1,s_2)} = id$. The bipartition $(\widetilde{\lambda}^2, \widetilde{\lambda}^1)$ can be defined from its symbol $\binom{\widetilde{L}_1}{\widetilde{L}_2}$ as follows.

Suppose first $s_2 \geq s_1$. Consider $x_1 = \min\{t \in L_1\}$. We associate to x_1 the integer $y_1 \in L_2$ such that

$$y_1 = \begin{cases} \max\{z \in L_2 \mid z \leq x_1\} \text{ if } \min\{z \in L_2\} \leq x_1, \\ \max\{z \in L_2\} \text{ otherwise.} \end{cases} \tag{7}$$

We repeat the same procedure to the lines $L_2 - \{y_1\}$ and $L_1 - \{x_1\}$. By induction this yields a sequence $\{y_1, ..., y_{d+1+s_1-s_2}\} \subset L_2$. Then we define $\widetilde{L}_1$ as the line obtained by reordering the integers of $\{y_1, ..., y_{d+1+s_2-s_1}\}$ and $\widetilde{L}_2$ as the line obtained by reordering the integers of $L_2 - \{y_1, ..., y_{d+1+s_1-s_2}\} + L_1$ (i.e. by reordering the set obtained by replacing in L_2 the entries $y_1, ..., y_{d+1+s_1-s_2}$ by those of L_1).

Now, suppose $s_2 < s_1$. Consider $x_1 = \min\{t \in L_2\}$. We associate to x_1 the integer $y_1 \in L_1$ such that

$$y_1 = \begin{cases} \min\{z \in L_1 \mid x_1 \leq z\} \text{ if } \max\{z \in L_1\} \geq x_1, \\ \min\{z \in L_1\} \text{ otherwise.} \end{cases} \tag{8}$$

We repeat the same procedure to the lines $L_1 - \{y_1\}$ and $L_2 - \{x_1\}$ and obtain a sequence $\{y_1, ..., y_{d+1+s_1-s_2}\} \subset L_1$. Then we define $\widetilde{L}_2$ as the line obtained by reordering the integers of $\{y_1, ..., y_{d+1+s_2-s_1}\}$ and $\widetilde{L}_1$ as the line obtained by reordering the integers of $L_1 - \{y_1, ..., y_{d+1+s_2-s_1}\} + L_2$.

Example 3.3. *Assume $(s_1, s_2) = (0, 3)$ and consider the bipartition of 38 given by $(\lambda^1, \lambda^2) = (6.5.5.4, 5.5.3.3.2)$. Then $d = 7$ and:*

$$S(\lambda^1, \lambda^2) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline -4+0 & -3+0 & -2+0 & -1+2 & 0+3 & 1+3 & 2+5 & 3+5 \\ \hline -4+0 & -3+4 & -2+5 & -1+5 & 0+6 \\ \cline{1-5} \end{array}$$

$$S(\lambda^1, \lambda^2) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline -4 & -3 & -2 & 1 & 3 & 4 & 7 & 8 \\ \hline -4 & 1 & 3 & 4 & 6 \\ \cline{1-5} \end{array}$$

We get $\{y_1, ..., y_5\} = \{-4, 1, 3, 4, -2\}$. This gives

$$S(\widetilde{\lambda}^2, \widetilde{\lambda}^1) = \begin{array}{|c|c|c|c|c|c|c|c|} \cline{1-5} -4 & -2 & 1 & 3 & 4 \\ \hline -4 & -3 & 1 & 3 & 4 & 6 & 7 & 8 \\ \hline \end{array}$$

and finally $(\widetilde{\lambda}^2, \widetilde{\lambda}^1) = (5.5.5.4.4.3, 4.4.3.1)$. Observe that both (λ^1, λ^2) and $(\widetilde{\lambda}^2, \widetilde{\lambda}^1)$ have rank equal to 38.

Let F be the flip involution on the set of bipartitions defined by $F(\mu^1, \mu^2) = (\mu^2, \mu^1)$. We introduce the map $\Phi^{\infty}_{(s_1,s_2)} = F \circ R^{\infty}_{(s_1,s_2)}$. With the previous notation, we have

$$\begin{array}{rccc} \Phi^{\infty}_{(s_1,s_2)} & : \Pi^2(n) & \to & \Pi^2(n) \\ & (\lambda^1, \lambda^2) & \mapsto & (\widetilde{\lambda}^1, \widetilde{\lambda}^2). \end{array}$$

By using the crystal isomorphism introduced in (4) and the properties of the R-matrix $R^{\infty}_{(s_1,s_2)}$ we have just recalled, we get the following proposition.

Proposition 3.4. *Assume that $l = 2$ and $e = \infty$. Consider $\mathbf{m}^+$ and $\mathbf{m}^-$ separated by the wall $\mathfrak{m}_{1,2}^{(s_1,s_2),\infty}$ such that $m^+ = (m_1^+, m_2^+)$ with $m_1^+ - s_1 - (m_2^+ - s_2) > 0$ and $m^- = (m_1^-, m_2^-)$ with $m_1^- - s_1 - (m_2^- - s_2) < 0$.*

1. *The map $\Phi^{\infty}_{(s_1,s_2)}$ defines a $\mathfrak{g}_\infty$-crystal isomorphism from $\mathcal{G}_{\infty,\mathbf{m}^+,\mathbf{s}}$ to $\mathcal{G}_{\infty,\mathbf{m}^-,\mathbf{s}}$.*
2. *The map $(\Phi^{\infty}_{(s_1,s_2)})^{-1} = R^{\infty}_{(s_2,s_1)} \circ F$ is a $\mathfrak{g}_\infty$-crystal isomorphism from $\mathcal{G}_{\infty,\mathbf{m}^-,\mathbf{s}}$ to $\mathcal{G}_{\infty,\mathbf{m}^+,\mathbf{s}}$.*
3. *The map $\Phi^{\infty}_{(s_1,s_2)}$ is the unique crystal isomorphism between $\mathcal{G}_{\infty,\mathbf{m}^+,\mathbf{s}}$ and $\mathcal{G}_{\infty,\mathbf{m}^-,\mathbf{s}}$ which preserves the rank of the bipartitions.*[1]

[1] More generally, this also shows there is only one graph isomorphism from $\mathcal{G}_{\infty,\mathbf{m}^+,\mathbf{s}}$ to $\mathcal{G}_{\infty,\mathbf{m}^-,\mathbf{s}}$ preserving both the labelling of the arrows and the rank of the bipartitions.

Assume that $l \in \mathbb{N}$. To construct a crystal isomorphism between two Fock spaces $\mathcal{G}_{\infty,\mathbf{m},\mathbf{s}}$ and $\mathcal{G}_{\infty,\mathbf{m}',\mathbf{s}}$ where $\mathbf{m}$ and $\mathbf{m}'$ are separated by the wall $\mathfrak{m} := \mathfrak{m}_{i,j}^{\mathbf{s},\infty}$ (and only by this wall), we define the map:

$$\begin{array}{rcl} \Phi_{\mathbf{m}\to\mathbf{m}'}^{\mathbf{s},\infty} \quad : \Pi^l(n) & \to & \Pi^l(n) \\ (\lambda^1,\dots,\lambda^l) & \mapsto & (\mu^1,\dots,\mu^l), \end{array}$$

such that $\mu^k = \lambda^k$ if $k \neq i,j$ and

$$(\mu^i,\mu^j) = \begin{cases} \Phi^{(s_i,s_j),\infty}(\lambda^i,\lambda^j) & \text{if} \quad m_i - s_i - (m_j - s_j) > 0, \\ (\Phi^{(s_i,s_j),\infty})^{-1}(\lambda^i,\lambda^j) & \text{otherwise.} \end{cases}$$

Proposition 3.5. $\Phi_{\mathbf{m}\to\mathbf{m}'}^{\mathbf{s},\infty}$ *defines a* $\mathfrak{g}_\infty$*-isomorphism of crystal between* $\mathcal{G}_{\infty,\mathbf{m},\mathbf{s}}$ *and* $\mathcal{G}_{\infty,\mathbf{m}',\mathbf{s}}$*.*

Proof. Set $\boldsymbol{\delta} = \mathbf{m} - \mathbf{s}$ and $\boldsymbol{\delta}' = \mathbf{m}' - \mathbf{s}$. By Proposition 3.1, since we can move in each chamber without changing the crystal structure we can assume that $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ have distinct coordinates. We can also assume that $\boldsymbol{m}$ and $\boldsymbol{m}'$ are very close to each other but belong to distinct half-spaces defined by the wall $\mathfrak{m}$. So the coordinates of $\boldsymbol{\delta} - \boldsymbol{\delta}'$ are small and we have $\delta_i - \delta_j > 0$ and $\delta_i' - \delta_j' < 0$ (or $\delta_i - \delta_j < 0$ and $\delta_i' - \delta_j' > 0$). We will assume $\delta_i - \delta_j > 0$ and $\delta_i' - \delta_j' < 0$, the arguments being analogous when $\delta_i - \delta_j < 0$ and $\delta_i' - \delta_j' > 0$.

Let σ be the permutation of $\{1,\dots,l\}$ corresponding to the decreasing reordering of $\boldsymbol{\delta}$. One can then choose the coordinates of $\boldsymbol{\delta} - \boldsymbol{\delta}'$ sufficiently small so that the permutation of $\{1,\dots,l\}$ yielding the decreasing reordering of $\boldsymbol{\delta}'$ becomes $\sigma' = \sigma \circ (i,j)$. We then have $\Phi_{\mathbf{m}\to\mathbf{m}'}^{\mathbf{s},\infty} = (\sigma')^{-1} \circ R_{(s_i,s_j)}^{\infty} \otimes Id^{l-2} \circ \sigma$ where we also denote by σ and σ' the crystal isomorphims of type (4) mapping $\mathcal{G}_{\infty,\mathbf{m},\mathbf{s}}$ and $\mathcal{G}_{\infty,\mathbf{m}',\mathbf{s}}$ on the JMMO structures $\mathcal{G}_{\infty,\sigma(\mathbf{s})}$ and $\mathcal{G}_{\infty,\sigma'(\mathbf{s})}$, respectively. Therefore, $\Phi_{\mathbf{m}\to\mathbf{m}'}^{\mathbf{s},\infty}$ is also a crystal isomorphism between the Fock spaces $\mathcal{G}_{\infty,\mathbf{m},\mathbf{s}}$ and $\mathcal{G}_{\infty,\mathbf{m}',\mathbf{s}}$. □

We now describe the canonical $\mathfrak{g}_e$-isomorphisms when $e \in \mathbb{N}$ is finite. Consider a wall:

$$\mathfrak{m}_{i,j,N}^{\mathbf{s},e} := \{(m_1,\dots,m_l) \,|s_i - m_i - (s_j - m_j) = N.e\}.$$

Assume that $\mathbf{m}$ and $\mathbf{m}'$ are separated by this wall (and only it) and that:

(9) $\qquad s_i - m_i - (s_j - m_j) > N.e$ and $s_i - m_i' - (s_j - m_j') < N.e.$

To define a $\mathfrak{g}_e$-crystal isomorphism between the crystal $\mathcal{G}_{e,\mathbf{m},\mathbf{s}}$ and $\mathcal{G}_{e,\mathbf{m}',\mathbf{s}}$, let us set $\widetilde{\mathbf{s}} = (s_1,\dots,s_i - N.e,\dots,s_l)$. Then an element $\widetilde{\mathbf{m}}$ is in the wall $\mathfrak{m}_{i,j,N}^{\mathbf{s},e}$ if and only if $\widetilde{\mathbf{m}}$ is in the set

$$\mathfrak{m}^\infty := \mathfrak{m}_{i,j}^{\infty,\widetilde{\mathbf{s}}} := \{(m_1,\dots,m_l) \,|(s_i - N.e) - m_i - (s_j - m_j) = 0\},$$

where $\widetilde{\mathbf{s}} = (s_1,s_2,\dots,s_i - Ne,\dots,s_l)$. The set $\mathfrak{m}_{i,j}^{\infty,\widetilde{\mathbf{s}}}$ is a wall for $\widetilde{\mathbf{s}}$ in the case where $e = \infty$ and we known that the map $\Phi_{\mathbf{m}\to\mathbf{m}'}^{\widetilde{\mathbf{s}},\infty}$ of Proposition 3.5 is a $\mathfrak{g}_\infty$-crystal isomorphism between $\mathcal{G}_{\infty,\mathbf{m},\widetilde{\mathbf{s}}}$ and $\mathcal{G}_{\infty,\mathbf{m}',\widetilde{\mathbf{s}}}$.

Theorem 3.6. *$\Phi^{\widetilde{\mathbf{s}},\infty}_{\mathbf{m}\to\mathbf{m}'}$ is a $\mathfrak{g}_e$-crystal isomorphism between $\mathcal{G}_{e,\mathbf{m},\mathbf{s}}$ and $\mathcal{G}_{e,\mathbf{m}',\mathbf{s}}$.*

Proof. Let define σ and σ' from Set $\boldsymbol{\delta} = \mathbf{m} - \widetilde{\mathbf{s}}$ and $\boldsymbol{\delta}' = \mathbf{m}' - \widetilde{\mathbf{s}}$ as we did in the previous proof. By Proposition 5.2.2 of [6] $R^{\infty}_{(\widetilde{s}_i,\widetilde{s}_j)} \otimes I^{l-2}$ is a $\mathfrak{g}_e$-crystal isomorphism between the JMMO structures $\mathcal{G}_{\infty,\sigma(\widetilde{\mathbf{s}})}$ and $\mathcal{G}_{\infty,\sigma'(\widetilde{\mathbf{s}})}$. Since we have $\Phi^{\mathbf{s},\infty}_{\mathbf{m}\to\mathbf{m}'} = (\sigma')^{-1} \circ R^{\infty}_{(\widetilde{s}_i,\widetilde{s}_j)} \otimes Id^{l-2} \circ \sigma$, we get that that $\Phi^{\widetilde{\mathbf{s}},\infty}_{\mathbf{m}\to\mathbf{m}'}$ is a $\mathfrak{g}_e$-crystal isomorphism between $\mathcal{G}_{e,\mathbf{m},\widetilde{\mathbf{s}}}$ and $\mathcal{G}_{e,\mathbf{m}',\widetilde{\mathbf{s}}}$ (see (4)). Now note that $\mathbf{s}$ and $\widetilde{\mathbf{s}}$ are two multicharges that are equivalent modulo e. So we have in fact the equalities $\mathcal{G}_{e,\mathbf{m},\widetilde{\mathbf{s}}} = \mathcal{G}_{e,\mathbf{m},\mathbf{s}}$ and $\mathcal{G}_{e,\mathbf{m}',\widetilde{\mathbf{s}}} = \mathcal{G}_{e,\mathbf{m}',\mathbf{s}}$ (see Remark 2.4). The result follows. □

To simplify, we will denote by $\Psi^{\mathbf{s}}_{\mathbf{m}\to\mathbf{m}'}$ the above bijection obtained by crossing a single wall when $\mathbf{m}$ and $\mathbf{m}'$ satisfy (9). The above result allows to compute certain $\mathfrak{g}_e$-crystal isomorphisms between two arbitrary Fock spaces $\mathcal{F}_{e,\mathbf{m},\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{m}',\mathbf{s}}$ by composing the $\mathfrak{g}_\infty$-crystal isomorphisms of Theorem 3.6.

Remark 3.7. *In* [7]*, we gave a simple criterion deciding whether a multipartition $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ is a highest weight vertex in a Uglov crystal structure. Using the discussion in §2.4, this criterion can be easy adapted to the crystal structure $\mathcal{G}_{e,\mathbf{m},\mathbf{s}}$.*

Example 3.8. *We will here consider the same example as* [9, ex 5.6]*, We assume that $\mathbf{s} = (0,0)$ and $e = 2$. We take $n = 3$. Then we have 3 hyperplanes to consider :*

$$\mathfrak{m}^{(0,0),2}_{1,2,-1} := \{(m_1,m_2) \in \mathbb{Q}^2 \mid m_2 - m_1 = -2\},$$

$$\mathfrak{m}^{(0,0),2}_{1,2,0} := \{(m_1,m_2) \in \mathbb{Q}^2 \mid m_2 - m_1 = 0\},$$

$$\mathfrak{m}^{(0,0),2}_{1,2,1} := \{(m_1,m_2) \in \mathbb{Q}^2 \mid m_2 - m_1 = 2\}.$$

The 2-partitions of 3 are :

$$(\emptyset,1.1.1),(\emptyset,2.1),(\emptyset,3),(1,1.1),(1,2),(1.1,1),(1.1.1,\emptyset),(2,1),(2.1,\emptyset),(3,\emptyset)$$

Now we pick one parameter in each of the four associated chambers :

(1) *Take $\mathbf{m}[1] = (m_1,m_2) \in \mathbb{Q}^2$ such that $m_2 - m_1 < -2$.*
(2) *Take $\mathbf{m}[2] = (m_1,m_2) \in \mathbb{Q}^2$ such that $0 > m_2 - m_1 > -2$.*
(3) *Take $\mathbf{m}[3] = (m_1,m_2) \in \mathbb{Q}^2$ such that $2 > m_2 - m_1 > 0$.*
(4) *Take $\mathbf{m}[4] = (m_1,m_2) \in \mathbb{Q}^2$ such that $m_2 - m_1 > 2$.*

The following table gives the isomorphisms computed using our procedure, the notation $(\star)$ indicates that the associated bipartitions are highest weight *vertices for the parameters considered.*

	$\mathbf{m}[1]$	$\mathbf{m}[2]$	$\mathbf{m}[3]$	$\mathbf{m}[4]$
$(\star)$	$(\emptyset, 1.1.1)$	$(\emptyset, 1.1.1)$	$(1.1.1, \emptyset)$	$(1.1.1, \emptyset)$
	$(\emptyset, 2.1)$	$(\emptyset, 2.1)$	$(2.1, \emptyset)$	$(2.1, \emptyset)$
$(\star)$	$(\emptyset, 3)$	$(1.1.1, \emptyset)$	$(\emptyset, 1.1.1)$	$(3, \emptyset)$
	$(1, 1.1)$	$(1, 1.1)$	$(1.1, 1)$	$(1.1, 1)$
	$(1, 2)$	$(\emptyset, 3)$	$(3, \emptyset)$	$(2, 1)$
$(\star)$	$(1.1, 1)$	$(1, 2)$	$(2, 1)$	$(1, 1.1)$
	$(1.1.1, \emptyset)$	$(1.1, 1)$	$(1, 1.1)$	$(\emptyset, 1.1.1)$
	$(2, 1)$	$(2, 1)$	$(1, 2)$	$(1, 2)$
	$(2.1, \emptyset)$	$(2.1, \emptyset)$	$(\emptyset, 2.1)$	$(\emptyset, 2.1)$
	$(3, \emptyset)$	$(3, \emptyset)$	$(\emptyset, 3)$	$(\emptyset, 3)$

4. Wall crossing bijections for Cherednik algebras

We now give an interpretation of the above isomorphisms in the context of rational Cherednik algebras. In [9], Losev has introduced certain combinatorial maps between the sets of simple modules in categories $\mathcal{O}$ for rational Cherednik algebras corresponding to different parameter values. We are going to explain how these maps (called "wall-crossing bijections") are connected with our crystal isomorphisms. We refer to [10] for more details on the representation theory of Cherednik algebras and for problems we are interested in in this section.

4.1. **Rational Cherednik algebras.** Let $\mathcal{H}_{\kappa,\mathbf{s}}(n)$ be the rational Cherednik algebra associated with the complex reflection group of type $W := G(l,1,n)$ acting on $\mathfrak{h} := \mathbb{C}^n$. As a vector space, this algebra is $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h})$ (where $S(V)$ denotes the symmetric algebra of the vector space V). It admits a presentation by generators and relations for which we refer to [5, §2.3].

Importantly, this presentation depends on a parameter $s := (\kappa, \mathbf{s}) \in \mathbb{C} \times \mathbb{C}^l$. This parameter is the one used in [9, 10] as well as in [5] (the reader may look at the relations between the different parametrizations given in [5, §2.3.2])

We will consider the category $\mathcal{O}_{n,s}$ for this algebra whose simple objects are parametrized by the set $\Pi^l(n)$, which also indexes the set of irreducible representations of the complex reflection group W in characteristic 0.

Remark 4.1. *An important problem in this theory is to compute the support of a simple module in the category $\mathcal{O}_{n,s}$ parametrized by an l-partition $\boldsymbol{\lambda}$. This in particular leads to a classification of the finite-dimensional simple modules in this category.*

As explained in [9, §4] and [10, §3.1.3], in most of the questions relative to the study of the category $\mathcal{O}$ we can assume (and we will do in the sequel) the following condition is satisfied.

Condition 4.2. *In the rest of the paper we assume that:*

(1) $\kappa = \frac{r}{e}$ *is a positive rational number where* r *and* e *are relatively prime,*
(2) $r.s_j \in \mathbb{Z}$ *for any* $j = 1, \dots, l$.

In particular, we have now $s := (\kappa, \mathbf{s}) \in \mathbb{Q} \times \mathbb{Q}^l$. Let us denote by $\mathcal{S}$ the subset of $\mathbb{Q} \times \mathbb{Q}^l$ satisfying (1) and (2).

In [11], Shan has introduced an action of the quantum group $\mathfrak{g}_e$ on a Fock space defined from a categorical action on the direct sum over n of the categories $\mathcal{O}_{n,s}$. This action heavily depends on the choice of the parameter s. It also induces a structure of $\mathfrak{g}_e$-crystal on the set of l-partitions. This crystal structure can be defined by using a relevant total order $\leq$ on z-nodes as we did in Section 2.2. Consider $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ two such z-nodes.

Definition 4.3. *We set* $\gamma \leq \gamma'$ *if and only if*

$$\kappa l(b - a + s_c) - c \leq \kappa l(b' - a' + s_{c'}) - c'.$$

The associated oriented graph is denoted by $\mathcal{G}_s$ *with indexing set* I_s.

Remark 4.4. *The order* $\leq$ *is the reverse of the one used in* [9] *where the crystal operators are also slightly defined differently. Nevertheless, the graph structures obtained are the same in both papers.*

Observe this is indeed a total order since the equality

$$\kappa l(b - a + s_c) - c = \kappa l(b' - a' + s_{c'}) - c',$$

implies that l divides $c - c'$. But $0 \leq |c - c'| < l$ so we have in fact $c = c'$ and $\gamma = \gamma'$. We are going to see that the graph structure $\mathcal{G}_s$ coincides with a graph structure already defined which is a crystal structure up to reparametrization of the colors. Since r and e are relatively prime, we have $e\mathbb{Z} + r\mathbb{Z} = \mathbb{Z}$ and $e\mathbb{Z} \cap r\mathbb{Z} = er\mathbb{Z}$. Then, for any integer a there exists a unique pair (c, d) such that $a = ed + rc$ and $c \in \{0, \dots, e-1\}$. Since we have $rs_j \in \mathbb{Z}$ for any $j = 1, \dots, l$, we can set:

$$rs_j = ed_j + rc_j, \tag{10}$$

where $c_j \in \{0, \dots, e-1\}$. Note then that c_j is equivalent to s_j modulo $\kappa^{-1}\mathbb{Z}$. Recall Definition 2.2. We have the following elementary lemma.

Lemma 4.5. *The map*

$$\begin{array}{rccc} \psi : & \mathbb{Z}/e\mathbb{Z} & \to & I_s \\ & i(mod\ e) & \mapsto & i(mod\ \kappa^{-1}\mathbb{Z}) \end{array}$$

is well defined and is a bijection

Proof. Since $e = r\kappa^{-1}$, we have $e\mathbb{Z} \subset \kappa^{-1}\mathbb{Z}$. This implies that we have a natural map from $\mathbb{Z}/e\mathbb{Z}$ to $\mathbb{Q}/\kappa^{-1}\mathbb{Z}$. By the comment preceding the lemma, this map is surjective onto I_s. As $\mathbb{Z} \cap \kappa^{-1}\mathbb{Z} = e\mathbb{Z}$, it is injective. This concludes the proof.

□

For $j = 1, \dots, l$ write also

$$m_j = s_j - \frac{j}{\kappa l} = s_j - \frac{je}{rl} \in \mathbb{Q},$$

and set $\boldsymbol{m} = (m_1, \dots, m_l) \in \mathbb{Q}^l$.

4.2. **Relation with extended JMMO Fock space structure (2).** In the following proposition, we will consider the oriented graph $\mathcal{G}_s$ with indexing set I_s and the crystal graph $\mathcal{G}_{e,\mathbf{m},\mathbf{c}}$ which indexing set $\mathbb{Z}/e\mathbb{Z}$ where $\mathbf{c} = (c_1, \dots, c_l)$ (see (10)).

Proposition 4.6. *The colored oriented graph $\mathcal{G}_s$ is equivalent to the crystal $\mathcal{G}_{e,\mathbf{m},\mathbf{c}}$. More precisely, we have an isomorphism of oriented graphs θ between $\mathcal{G}_{e,\mathbf{m},\mathbf{c}}$ and $\mathcal{G}_s$ and according to the notation of §2.2, the following maps :*

$$\begin{array}{cccc} \Psi & \Pi^l(n) & \to & \Pi^l(n) \\ & \boldsymbol{\lambda} & \mapsto & \boldsymbol{\lambda} \end{array}, \quad \begin{array}{cccc} \psi & \mathbb{Z}/e\mathbb{Z} & \to & I_s \\ & i(mod\ e) & \mapsto & i(mod\ \kappa^{-1}\mathbb{Z}). \end{array}$$

Proof. First, we show that two boxes have the same residue for $(\kappa, \mathbf{s})$ if and only they have the same residue for $(1/e, \mathbf{c})$. This follows from the fact that, for two nodes (x, y, i) and (x', y', j) of an l-partition, we have

$$x - y + s_i = x' - y' + s_j + \kappa^{-1}\mathbb{Z}$$

if and only if

$$r(x - y + s_i) = r(x' + y' + s_j) + e\mathbb{Z},$$

because $\kappa = \frac{r}{e}$. By using (10) we get

$$r(x - y + c_i) = r(x' + y' + c_j) + e\mathbb{Z}.$$

Now both $(x - y + c_i)$ and $(x' + y' + c_j)$ are integers. As $(e, r) = 1$, we have

$$x - y + c_i = x' + y' + c_j + e\mathbb{Z},$$

which is what we wanted to show. Observe also that the residue of (x, y, i) for $(1/e, \mathbf{c})$ and $(\kappa, \mathbf{s})$ are respectively equal to

$$(x - y + c_i)(\text{mod}e) \text{ and } (x - y + s_i)(\text{mod}\kappa^{-1}\mathbb{Z}),$$

and we have

$$(x - y + c_i) \equiv (x - y + s_i)(\text{mod}\kappa^{-1}\mathbb{Z}).$$

It just remains to show that the order $\leq$ on z-nodes corresponds to the order $\preceq_{\mathbf{m}}$.

Let $\gamma = (x, y, i)$ and $\gamma' = (x', y', j)$ be two nodes of the l-partition. Then we have the equivalences

$$\begin{aligned} \gamma \leq \gamma' \Longleftrightarrow \kappa(x - y + s_c) - \frac{c}{l} &= \kappa(x' - y' + s_{c'}) - \frac{c'}{l} \\ &\Longleftrightarrow x - y + m_i \leq x' - y' + m_j. \end{aligned}$$

which is exactly what we wanted. □

Remark 4.7. *Combining this proposition with Remark 3.7, we obtain a criterion to decide whether an l-partition is a highest weight in $\mathcal{G}_{\mathbf{s}}$. The finite-dimensional simple modules in the associated category $\mathcal{O}$ of the Cherednik algebra are in particular parametrized by l-partitions satisfying this criterion. However it is not a sufficient condition for being finite-dimensional. To obtain a necessary and sufficient condition, one has to take into account the action of the Heisenberg algebra. We refer to* [4] *for other results in this direction and a detailed investigation of the crystal action of the Heisenberg algebra on $\mathcal{G}_{\mathbf{s}}$.*

Remark 4.8. *One can also combine the results of* [9] *together with the approach in* [6]*. Given a parameter s, the crystal structure $\mathcal{G}_s$ is equivalent to a crystal $\mathcal{G}_{e,\mathbf{m},\mathbf{c}}$ (by the previous proposition) which is isomorphic to a JMMO crystal $\mathcal{G}_{e,\mathbf{s}}$ (by §2.4). Now we described in* [6] *the crystal isomorphisms on the JMMO crystals $\mathcal{G}_{e,\mathbf{s}}$, $s \in \mathbb{Z}^l$ corresponding to the action of the extended affine symmetric group on $\mathbb{Z}^l$. This eventually leads to a rank preserving bijection between $\mathcal{G}_s$ and a crystal structure $\mathcal{G}_{s'}$ associated to an asymptotic parameter where the support can be calculated thanks to the result in* [9]*. As all the rank-preserving bijections involved here leave the support invariant, this gives another way to compute the support of a simple module by reduction to the asymptotic case.*

Let us now review the wall crossing bijections. Fix $n \in \mathbb{N}$. In the set of parameters $\mathcal{S}$, one can define certain hyperplanes called essential walls. Set

$$h_j = \kappa m_j = \kappa s_j - j/l, \ j = 1, \dots, l.$$

Definition 4.9. *Recall n is fixed. Given i, j distinct in $\{1, \dots, l\}$, an essential wall parametrized by (i, j) is a set of parameters $s = (\kappa, \mathbf{s}) \in \mathcal{S}$ such that there exists an integer a satisfying:*

(1) $|a| < n$,
(2) $m_i - m_j = a$,
(3) $s_i - s_j - a \in \kappa^{-1}\mathbb{Z}$.

Consider two parameters $s = (\kappa, \mathbf{s})$ and $s' = (\kappa', \mathbf{s}')$ in $\mathcal{S}$ satisfying :

$$\kappa - \kappa' \in \mathbb{Z} \text{ and } \forall j \in \{1, \dots, l\} \ \kappa s_j - \kappa' s'_j \in \mathbb{Z} \tag{11}$$

and such that s and s' are separated by one essential wall.

Remark 4.10. *Assume s and s' satisfy (11). This means that there exist $k \in \mathbb{Z}$ and $t \in \mathbb{Z}$ such that $r's'_j = rs_j + ke$ and $r' = r + t.e$ (here $\kappa' = \frac{r'}{e}$).*

By (10), for each $j = 1, \dots, l$, there exists a unique $(c_j, c'_j) \in \{0, \dots, e-1\}^2$ and $(d_j, d'_j) \in \mathbb{Z}^2$ such that

$$rs_j = ed_j + rc_j, \ r's'_j = ed'_j + r'c'_j.$$

We obtain:

$$\begin{array}{rcl} r's'_j & = & rs_j + ke \\ & = & ed_j + rc_j + ke \\ & = & ed_j + r'c_j - tec_j + ke \\ & = & e(d_j - te + k) + r'c_j. \end{array}$$

We thus have $c'_j = c_j$ and $d'_j = d_j - te + k$.

By Proposition 4.6 $\mathcal{G}_s$ and $\mathcal{G}_{s'}$ are respectively isomorphic to $\mathcal{G}_{e,\mathbf{m},\mathbf{c}}$ and $\mathcal{G}_{e,\mathbf{m}',\mathbf{c}}$ for good choices of $\mathbf{m}$ and $\mathbf{m}'$. But $\mathcal{G}_{e,\mathbf{m}',\mathbf{c}}$ and $\mathcal{G}_{e,\mathbf{m},\mathbf{c}}$ are isomorphic as crystal graphs because the associated Fock spaces have the same multi-charge. So $\mathcal{G}_{\mathbf{s}}$ and $\mathcal{G}_{\mathbf{s}'}$ are isomorphic as soon as s and s' satisfy (11).

In [9, Prop. 5.9], Losev defines a bijection $\mathfrak{wc}_{s\to s'}$ between l-partitions which is an isomorphism of graphs in the sense of §2.2.

Proposition 4.11. *Given s and s' in $\mathcal{S}$ satisfying (11) and separated by the essential wall parametrized by (i,j), there is a bijection $\mathfrak{wc}_{s\to s'}$ on the set of l-partitions such that if $\boldsymbol{\mu} = \mathfrak{wc}_{s\to s'}(\boldsymbol{\lambda})$ we have*

- $\lambda^k = \mu^k$ *if* $k \neq i$ *and* $k \neq j$,
- $(\mu^i, \mu^j) = \mathfrak{wc}_{(\kappa,s_i,s_j)\to(\kappa',s'_i,s'_j)}(\lambda^i, \lambda^j)$

where $\mathfrak{wc}_{(\kappa,s_i,s_j)\to(\kappa',s'_i,s'_j)}$ is the unique graph isomorphism between $\mathcal{G}_s$ and $\mathcal{G}_{s'}$ preserving both the labelling of the arrows and the rank of the bipartitions.

Let us look at the essential hyperplanes and check that they are the same as the ones define in §2.3. Assume that we are in an essential hyperplane. This implies that there exists $a \in \mathbb{Z}$ such that

$$s_i - s_j - a \in \kappa^{-1}\mathbb{Z} \qquad \text{and} \qquad m_i - m_j = a.$$

Thus there exists $x \in \mathbb{Z}$ such that $s_i - s_j = a + \kappa^{-1}x$. We obtain:

$$e(d_i - d_j - x) = r(c_j - c_i + a).$$

This implies that $c_j - c_i = -a + e\mathbb{Z}$ because e and r are relatively prime. Thus, we have :

$$c_j - m_j - (c_i - m_i) \in e\mathbb{Z}.$$

Conversely, assume we have $c_j - m_j - (c_i - m_i) = eN$ with $N \in \mathbb{Z}$ with the above relation between the parameters. We obtain that

$$m_i - m_j = c_i - c_j + eN.$$

Let us then set $a = c_i - c_j + eN$ which is in $\mathbb{Z}$. Now by (10) we have $s_i = \kappa^{-1}d_i + c_i$ and $s_j = \kappa^{-1}d_j + c_j$. We thus get

$$s_i - s_j - a = \kappa^{-1}(d_i - d_j) + (c_i - c_j) - a = \kappa^{-1}(d_i - d_j + rN) \in \kappa^{-1}\mathbb{Z}.$$

We now want to check that the wall crossing bijections correspond to our bijections defined in §3.2.

Theorem 4.12. *The wall crossing bijection $\mathfrak{wc}_{s\to s'}$ corresponds to $\Psi^{\mathbf{c}}_{\mathbf{m}\to\mathbf{m}'}$.*

Proof. Since both $\mathfrak{wc}_{s\to s'}$ and $\Psi^{\mathbf{c}}_{\mathbf{m}\to\mathbf{m}'}$ modify the component of indices i and j in the l-partition it suffices to show that for any bipartition (λ^1,λ^2) we have $\mathfrak{wc}_{(\kappa,s_i,s_j)\to(\kappa',s'_i,s'_j)}(\lambda^i,\lambda^j)=\Psi^{(c_i,c_j)}_{(m_i,m_j)\to(m'_i,m'_j)}(\lambda^i,\lambda^j)$. But $\mathfrak{wc}_{(\kappa,s_i,s_j)\to(\kappa',s'_i,s'_j)}$ and $\Psi^{(c_i,c_j)}_{(m_i,m_j)\to(m'_i,m'_j)}$ are graph isomorphisms which preserve the labelling of the arrows and the rank of the bipartitions. Since there is only one such isomorphism, they coincide. □

Remark 4.13. *Assume now $s:=(\kappa,\mathbf{s})$ where κ is a rational negative number. To each l-partition $\boldsymbol{\lambda}=(\lambda^1,\ldots,\lambda^l)$ we associate its conjugate $\boldsymbol{\lambda}^{\#}=((\lambda^l)^{\#},\ldots,(\lambda^1)^{\#})$ where $(\lambda^c)^{\#}$ is the conjugate of the partition λ^c for any $k=1,\ldots,l$. There is a natural bijection between the nodes of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^{\#}$ which maps $\gamma=(a,b,c)\in\boldsymbol{\lambda}$ on $\gamma^{\#}=(b,a,l-c)$. Set $s^{\#}:=(-\kappa,\mathbf{s}^{\#})$ where $\mathbf{s}^{\#}=(-s_l,\ldots,-s_1)$. The map $\#$ then defines a graph isomorphism twisted-equivariant for the labelling from the graphs $\mathcal{G}_s$ to $\mathcal{G}_{s^{\#}}$ which sends each l-partition $\boldsymbol{\lambda}$ on $\boldsymbol{\lambda}^{\#}$ and each arrow $\boldsymbol{\lambda}\overset{z}{\to}\boldsymbol{\mu}$ on $\boldsymbol{\lambda}^{\#}\overset{-z}{\to}\boldsymbol{\mu}^{\#}$. This implies that the wall crossing maps for κ and $-\kappa$ coincide up to conjugation by $\#$.*

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